Problem 3.42

Coherent states of the harmonic oscillator. Among the stationary states of the harmonic oscillator (Equation 2.68) only $n = 0$ hits the uncertainty limit ($\sigma_x\sigma_p = \hbar/2$); in general, $\sigma_x\sigma_p = (2n+1)\hbar/2$, as you found in Problem 2.12. But certain *linear combinations* (known as **coherent states**) also minimize the uncertainty product. They are (as it turns out) *eigenfunctions of the lowering operator:*[42]

$$a_-|\alpha\rangle = \alpha|\alpha\rangle$$

(the eigenvalue α can be any complex number).

(a) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ in the state $|\alpha\rangle$. *Hint:* Use the technique in Example 2.5, and remember that a_+ is the hermitian conjugate of a_-. Do *not* assume α is real.

(b) Find σ_x and σ_p; show that $\sigma_x\sigma_p = \hbar/2$.

(c) Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle.$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}}c_0.$$

(d) Determine c_0 by normalizing $|\alpha\rangle$. *Answer:* $\exp\left(-|\alpha|^2/2\right)$.

(e) Now put in the time dependence:

$$|n\rangle \to e^{-iE_nt/\hbar}\,|n\rangle,$$

and show that $|\alpha(t)\rangle$ remains an eigenstate of a_-, but the eigen*value* evolves in time:

$$\alpha(t) = e^{-i\omega t}\alpha.$$

So a coherent state *stays* coherent, and continues to minimize the uncertainty product.

(f) Based on your answers to (a), (b), and (e), find $\langle x\rangle$ and σ_x as functions of time. It helps if you write the complex number α as

$$\alpha = C\sqrt{\frac{m\omega}{2\hbar}}e^{i\phi}$$

for real numbers C and ϕ. *Comment:* In a sense, coherent states behave quasi-classically.

(g) Is the ground state ($|n = 0\rangle$) itself a coherent state? If so, what is the eigenvalue?

Solution

[42]There are no normalizable eigenfunctions of the *raising* operator.

Part (a)

In Example 2.5 on page 47 both the position and momentum operators are written in terms of the promotion and demotion operators, $\hat{a}_+$ and $\hat{a}_-$, respectively.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-) \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

Calculate the expectation value of x in the state $|\alpha\rangle$.

$$\begin{aligned}
\langle x\rangle &= \langle \alpha \,|\, \hat{x} \,|\, \alpha\rangle \\
&= \langle\alpha| \cdot (\hat{x}|\alpha\rangle) \\
&= \langle\alpha| \cdot \left[\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)|\alpha\rangle\right] \\
&= \langle\alpha| \cdot \sqrt{\frac{\hbar}{2m\omega}}\Big(\hat{a}_+|\alpha\rangle + \hat{a}_-|\alpha\rangle\Big) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\Big[\langle\alpha| \cdot (\hat{a}_+|\alpha\rangle) + \langle\alpha| \cdot (\hat{a}_-|\alpha\rangle)\Big] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left\{\Big[(\hat{a}_+|\alpha\rangle)^\dagger \cdot (\langle\alpha|)^\dagger\Big]^* + \langle\alpha| \cdot (\hat{a}_-|\alpha\rangle)\right\} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left\{\Big[(\langle\alpha|\hat{a}_+^\dagger) \cdot |\alpha\rangle\Big]^* + \langle\alpha| \cdot (\hat{a}_-|\alpha\rangle)\right\} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left\{\Big[(\langle\alpha|\hat{a}_-) \cdot |\alpha\rangle\Big]^* + \langle\alpha| \cdot (\hat{a}_-|\alpha\rangle)\right\} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left\{\Big[\langle\alpha| \cdot (\hat{a}_-|\alpha\rangle)\Big]^* + \langle\alpha| \cdot (\hat{a}_-|\alpha\rangle)\right\} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left\{\Big[\langle\alpha| \cdot (\alpha|\alpha\rangle)\Big]^* + \langle\alpha| \cdot (\alpha|\alpha\rangle)\right\} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\Big(\alpha\langle\alpha \,|\, \alpha\rangle\Big)^* + \alpha\langle\alpha \,|\, \alpha\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\Big[(\alpha \cdot 1)^* + \alpha \cdot 1\Big] \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\alpha^* + \alpha) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(2\,\mathrm{Re}\ \alpha) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\ \alpha
\end{aligned}$$

Calculate the expectation value of x^2 in the state $|\alpha\rangle$. Recall that $[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$.

$$
\begin{aligned}
\langle x^2 \rangle &= \langle \alpha \,|\, \hat{x}^2 \,|\, \alpha \rangle \\
&= \langle \alpha | \cdot \big(\hat{x}^2 |\alpha\rangle \big) \\
&= \langle \alpha | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+ + \hat{a}_-)(\hat{a}_+ + \hat{a}_-)|\alpha\rangle \right] \\
&= \langle \alpha | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+\hat{a}_+ + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-)|\alpha\rangle \right] \\
&= \langle \alpha | \cdot \left[\frac{\hbar}{2m\omega} (\hat{a}_+\hat{a}_+ + 2\hat{a}_+\hat{a}_- + 1 + \hat{a}_-\hat{a}_-)|\alpha\rangle \right] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \Big[\hat{a}_+\hat{a}_+|\alpha\rangle + 2\hat{a}_+(\hat{a}_-|\alpha\rangle) + |\alpha\rangle + \hat{a}_-(\hat{a}_-|\alpha\rangle) \Big] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \Big[\hat{a}_+\hat{a}_+|\alpha\rangle + 2\hat{a}_+(\alpha|\alpha\rangle) + |\alpha\rangle + \hat{a}_-(\alpha|\alpha\rangle) \Big] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \Big[\hat{a}_+\hat{a}_+|\alpha\rangle + 2\alpha\hat{a}_+|\alpha\rangle + |\alpha\rangle + \alpha(\hat{a}_-|\alpha\rangle) \Big] \\
&= \langle \alpha | \cdot \frac{\hbar}{2m\omega} \Big[\hat{a}_+\hat{a}_+|\alpha\rangle + 2\alpha\hat{a}_+|\alpha\rangle + |\alpha\rangle + \alpha(\alpha|\alpha\rangle) \Big] \\
&= \frac{\hbar}{2m\omega} \Big[\langle \alpha | \cdot \hat{a}_+\hat{a}_+|\alpha\rangle + 2\alpha\langle \alpha | \cdot \hat{a}_+|\alpha\rangle + \langle \alpha \,|\, \alpha \rangle + \alpha^2 \langle \alpha \,|\, \alpha \rangle \Big] \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[(\hat{a}_+\hat{a}_+|\alpha\rangle)^\dagger \cdot (\langle \alpha |)^\dagger \Big]^* + 2\alpha \Big[(\hat{a}_+|\alpha\rangle)^\dagger \cdot (\langle \alpha |)^\dagger \Big]^* + 1 + \alpha^2 \cdot 1 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[\Big(\langle \alpha | \hat{a}_+^\dagger \hat{a}_+^\dagger \Big) \cdot |\alpha\rangle \Big]^* + 2\alpha \Big[\Big(\langle \alpha | \hat{a}_+^\dagger \Big) \cdot |\alpha\rangle \Big]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[(\langle \alpha | \hat{a}_-\hat{a}_-) \cdot |\alpha\rangle \Big]^* + 2\alpha \Big[(\langle \alpha | \hat{a}_-) \cdot |\alpha\rangle \Big]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[(\langle \alpha | \hat{a}_-) \cdot (\hat{a}_-|\alpha\rangle) \Big]^* + 2\alpha \Big[\langle \alpha | \cdot (\hat{a}_-|\alpha\rangle) \Big]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[(\langle \alpha | \hat{a}_-) \cdot (\alpha|\alpha\rangle) \Big]^* + 2\alpha \Big[\langle \alpha | \cdot (\alpha|\alpha\rangle) \Big]^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[\langle \alpha | \cdot \alpha\, (\hat{a}_-|\alpha\rangle) \Big]^* + 2\alpha \Big(\alpha \langle \alpha \,|\, \alpha \rangle \Big)^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left\{ \Big[\langle \alpha | \cdot \alpha\, (\alpha|\alpha\rangle) \Big]^* + 2\alpha \Big(\alpha \cdot 1 \Big)^* + 1 + \alpha^2 \right\} \\
&= \frac{\hbar}{2m\omega} \left[\Big(\alpha^2 \langle \alpha \,|\, \alpha \rangle \Big)^* + 2\alpha\alpha^* + 1 + \alpha^2 \right]
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\langle x^2\rangle &= \frac{\hbar}{2m\omega}\left[\left(\alpha^2\cdot 1\right)^* + 2\alpha\alpha^* + 1 + \alpha^2\right] \\
&= \frac{\hbar}{2m\omega}\left[(\alpha^*)^2 + 2\alpha^*\alpha + \alpha^2 + 1\right] \\
&= \frac{\hbar}{2m\omega}\left[(\alpha^* + \alpha)^2 + 1\right] \\
&= \frac{\hbar}{2m\omega}\left[(2\,\text{Re }\alpha)^2 + 1\right] \\
&= \frac{\hbar}{2m\omega}\left[4(\text{Re }\alpha)^2 + 1\right].
\end{aligned}$$

Calculate the expectation value of p in the state $|\alpha\rangle$.

$$\begin{aligned}
\langle p\rangle &= \langle \alpha \,|\, \hat{p} \,|\, \alpha\rangle \\
&= \langle\alpha|\cdot(\hat{p}|\alpha\rangle) \\
&= \langle\alpha|\cdot\left[i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)|\alpha\rangle\right] \\
&= \langle\alpha|\cdot i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_+|\alpha\rangle - \hat{a}_-|\alpha\rangle\right) \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle\alpha|\cdot(\hat{a}_+|\alpha\rangle) - \langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left\{\left[(\hat{a}_+|\alpha\rangle)^\dagger\cdot(\langle\alpha|)^\dagger\right]^* - \langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right\} \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left\{\left[(\langle\alpha|\hat{a}_+^\dagger)\cdot|\alpha\rangle\right]^* - \langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right\} \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left\{\left[(\langle\alpha|\hat{a}_-)\cdot|\alpha\rangle\right]^* - \langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right\} \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left\{\left[\langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right]^* - \langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right\} \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left\{\left[\langle\alpha|\cdot(\alpha|\alpha\rangle)\right]^* - \langle\alpha|\cdot(\alpha|\alpha\rangle)\right\} \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\left(\alpha\langle\alpha\,|\,\alpha\rangle\right)^* - \alpha\langle\alpha\,|\,\alpha\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[(\alpha\cdot 1)^* - \alpha\cdot 1\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}\langle p\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}(\alpha^* - \alpha)\\ &= 2\sqrt{\frac{\hbar m\omega}{2}}\left(\frac{\alpha-\alpha^*}{2i}\right)\\ &= \sqrt{2\hbar m\omega}\,\text{Im }\alpha.\end{aligned}$$

Calculate the expectation value of p^2 in the state $|\alpha\rangle$.

$$\begin{aligned}\langle p^2\rangle &= \langle \alpha \,|\, \hat{p}^2 \,|\, \alpha\rangle\\ &= \langle\alpha|\cdot\left(\hat{p}^2|\alpha\rangle\right)\\ &= \langle\alpha|\cdot\left[-\frac{\hbar m\omega}{2}(\hat{a}_+ - \hat{a}_-)(\hat{a}_+ - \hat{a}_-)|\alpha\rangle\right]\\ &= \langle\alpha|\cdot\left[-\frac{\hbar m\omega}{2}(\hat{a}_+\hat{a}_+ - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-)|\alpha\rangle\right]\\ &= \langle\alpha|\cdot\left[-\frac{\hbar m\omega}{2}(\hat{a}_+\hat{a}_+ - 2\hat{a}_+\hat{a}_- - 1 + \hat{a}_-\hat{a}_-)|\alpha\rangle\right]\\ &= \langle\alpha|\cdot\frac{-\hbar m\omega}{2}\left[\hat{a}_+\hat{a}_+|\alpha\rangle - 2\hat{a}_+(\hat{a}_-|\alpha\rangle) - |\alpha\rangle + \hat{a}_-(\hat{a}_-|\alpha\rangle)\right]\\ &= \langle\alpha|\cdot\frac{-\hbar m\omega}{2}\left[\hat{a}_+\hat{a}_+|\alpha\rangle - 2\hat{a}_+(\alpha|\alpha\rangle) - |\alpha\rangle + \hat{a}_-(\alpha|\alpha\rangle)\right]\\ &= \langle\alpha|\cdot\frac{-\hbar m\omega}{2}\left[\hat{a}_+\hat{a}_+|\alpha\rangle - 2\alpha\hat{a}_+|\alpha\rangle - |\alpha\rangle + \alpha(\hat{a}_-|\alpha\rangle)\right]\\ &= \langle\alpha|\cdot\frac{-\hbar m\omega}{2}\left[\hat{a}_+\hat{a}_+|\alpha\rangle - 2\alpha\hat{a}_+|\alpha\rangle - |\alpha\rangle + \alpha(\alpha|\alpha\rangle)\right]\\ &= -\frac{\hbar m\omega}{2}\left[\langle\alpha|\cdot\hat{a}_+\hat{a}_+|\alpha\rangle - 2\alpha\langle\alpha|\cdot\hat{a}_+|\alpha\rangle - \langle\alpha\,|\,\alpha\rangle + \alpha^2\langle\alpha\,|\,\alpha\rangle\right]\\ &= -\frac{\hbar m\omega}{2}\left\{\left[(\hat{a}_+\hat{a}_+|\alpha\rangle)^\dagger\cdot(\langle\alpha|)^\dagger\right]^* - 2\alpha\left[(\hat{a}_+|\alpha\rangle)^\dagger\cdot(\langle\alpha|)^\dagger\right]^* - 1 + \alpha^2\cdot 1\right\}\\ &= -\frac{\hbar m\omega}{2}\left\{\left[\left(\langle\alpha|\hat{a}_+^\dagger\hat{a}_+^\dagger\right)\cdot|\alpha\rangle\right]^* - 2\alpha\left[\left(\langle\alpha|\hat{a}_+^\dagger\right)\cdot|\alpha\rangle\right]^* - 1 + \alpha^2\right\}\\ &= -\frac{\hbar m\omega}{2}\left\{\left[(\langle\alpha|\hat{a}_-\hat{a}_-)\cdot|\alpha\rangle\right]^* - 2\alpha\left[(\langle\alpha|\hat{a}_-)\cdot|\alpha\rangle\right]^* - 1 + \alpha^2\right\}\\ &= -\frac{\hbar m\omega}{2}\left\{\left[(\langle\alpha|\hat{a}_-)\cdot(\hat{a}_-|\alpha\rangle)\right]^* - 2\alpha\left[\langle\alpha|\cdot(\hat{a}_-|\alpha\rangle)\right]^* - 1 + \alpha^2\right\}\\ &= -\frac{\hbar m\omega}{2}\left\{\left[(\langle\alpha|\hat{a}_-)\cdot(\alpha|\alpha\rangle)\right]^* - 2\alpha\left[\langle\alpha|\cdot(\alpha|\alpha\rangle)\right]^* - 1 + \alpha^2\right\}\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle p^2 \rangle &= -\frac{\hbar m\omega}{2}\left\{\Big[\langle\alpha|\cdot\alpha\left(\hat{a}_-|\alpha\rangle\right)\Big]^* - 2\alpha\left(\alpha\langle\alpha\,|\,\alpha\rangle\right)^* - 1 + \alpha^2\right\} \\
&= -\frac{\hbar m\omega}{2}\left\{\Big[\langle\alpha|\cdot\alpha\left(\alpha|\alpha\rangle\right)\Big]^* - 2\alpha\left(\alpha\cdot 1\right)^* - 1 + \alpha^2\right\} \\
&= -\frac{\hbar m\omega}{2}\left[\left(\alpha^2\langle\alpha\,|\,\alpha\rangle\right)^* - 2\alpha\alpha^* - 1 + \alpha^2\right] \\
&= -\frac{\hbar m\omega}{2}\left[\left(\alpha^2\cdot 1\right)^* - 2\alpha\alpha^* - 1 + \alpha^2\right] \\
&= -\frac{\hbar m\omega}{2}[(\alpha^*)^2 - 2\alpha^*\alpha + \alpha^2 - 1] \\
&= -\frac{\hbar m\omega}{2}[(\alpha^* - \alpha)^2 - 1] \\
&= -\frac{\hbar m\omega}{2}\left[-4\left(\frac{\alpha-\alpha^*}{2i}\right)^2 - 1\right] \\
&= \frac{\hbar m\omega}{2}\left[4(\text{Im }\alpha)^2 + 1\right].
\end{aligned}$$

Part (b)

Now that all the expectation values are known, the standard deviations in position and momentum can be evaluated.

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}\left[4(\text{Re }\alpha)^2 + 1\right] - \left(\sqrt{\frac{2\hbar}{m\omega}}\,\text{Re }\alpha\right)^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}\left[4(\text{Im }\alpha)^2 + 1\right] - \left(\sqrt{2\hbar m\omega}\,\text{Im }\alpha\right)^2} = \sqrt{\frac{\hbar m\omega}{2}}$$

Therefore, the uncertainty product for the state $|\alpha\rangle$ is

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}.$$

Part (c)

Expand the coherent state $|\alpha\rangle$ in terms of the energy eigenstates of the harmonic oscillator.

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

To solve for c_n, take the inner product of $|\alpha\rangle$ and $|q\rangle$, another energy eigenstate.

$$\begin{aligned}
\langle q \,|\, \alpha\rangle &= \langle q| \cdot \sum_{n=0}^{\infty} c_n |n\rangle \\
&= \sum_{n=0}^{\infty} c_n \langle q \,|\, n\rangle \\
&= \sum_{n=0}^{\infty} c_n \delta_{qn}
\end{aligned}$$

Because the eigenstates are orthonormal, every term in this infinite series vanishes except for one—the one for which $n = q$.

$$\langle n \,|\, \alpha\rangle = c_n$$

Recall that the nth eigenstate of the harmonic oscillator is obtained by applying the promotion operator to the ground state n times: $\psi_n(x) = A_n(\hat{a}_+)^n\psi_0(x)$, where A_n is a normalization constant.

$$\begin{aligned}
c_n &= \langle n \,|\, \alpha\rangle \\
&= \langle n| \cdot |\alpha\rangle \\
&= (|n\rangle)^\dagger \cdot |\alpha\rangle \\
&= \left[A_n(\hat{a}_+)^n|0\rangle\right]^\dagger \cdot |\alpha\rangle \\
&= A_n^* \left[\langle 0| \left(\hat{a}_+^\dagger\right)^n\right] \cdot |\alpha\rangle \\
&= A_n^* \left[\langle 0|(\hat{a}_-)^n\right] \cdot |\alpha\rangle \\
&= A_n^* \langle 0| \cdot \left[(\hat{a}_-)^n|\alpha\rangle\right] \\
&= A_n^* \langle 0| \cdot \left(\alpha^n|\alpha\rangle\right) \\
&= A_n^* \alpha^n \langle 0 \,|\, \alpha\rangle \\
&= A_n^* \alpha^n c_0
\end{aligned}$$

The aim now is to find the normalization constant A_n. Since the energy of the nth eigenstate is $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, the TISE for the harmonic oscillator becomes

$$\hat{H}|n\rangle = E_n|n\rangle \quad \Rightarrow \quad \begin{cases} \hbar\omega\left(\hat{a}_-\hat{a}_+ - \dfrac{1}{2}\right)|n\rangle = \left(n + \dfrac{1}{2}\right)\hbar\omega|n\rangle \\ \hbar\omega\left(\hat{a}_+\hat{a}_- + \dfrac{1}{2}\right)|n\rangle = \left(n + \dfrac{1}{2}\right)\hbar\omega|n\rangle \end{cases} \quad \rightarrow \quad \begin{cases} \hat{a}_-\hat{a}_+|n\rangle = (n+1)|n\rangle \\ \hat{a}_+\hat{a}_-|n\rangle = n|n\rangle \end{cases}.$$

Suppose that the promotion and demotion operators satisfy

$$\hat{a}_+|n\rangle = f_n|n+1\rangle \quad \text{and} \quad \hat{a}_-|n\rangle = g_n|n-1\rangle.$$

Then

$$\begin{aligned} (\langle n|\hat{a}_-)\cdot(\hat{a}_+|n\rangle) = \langle n|\hat{a}_-\hat{a}_+|n\rangle &= \langle n|\cdot(\hat{a}_-\hat{a}_+|n\rangle) \\ \left(\hat{a}_-^\dagger|n\rangle\right)^\dagger\cdot(\hat{a}_+|n\rangle) = \qquad\qquad &= \langle n|\cdot[(n+1)|n\rangle] \\ (\hat{a}_+|n\rangle)^\dagger\cdot(\hat{a}_+|n\rangle) = \qquad\qquad &= (n+1)\langle n\,|\,n\rangle \\ (f_n|n+1\rangle)^\dagger\cdot(f_n|n+1\rangle) = \qquad\qquad &= (n+1)\cdot 1 \\ (f_n^*\langle n+1|)\cdot(f_n|n+1\rangle) = \qquad\qquad &= n+1 \\ f_n^* f_n\langle n+1\,|\,n+1\rangle = \qquad\qquad & \\ |f_n|^2\cdot 1 = \qquad\qquad & \\ |f_n|^2 = \qquad\qquad & \end{aligned}$$

and

$$\begin{aligned} (\langle n|\hat{a}_+)\cdot(\hat{a}_-|n\rangle) = \langle n|\hat{a}_+\hat{a}_-|n\rangle &= \langle n|\cdot(\hat{a}_+\hat{a}_-|n\rangle) \\ \left(\hat{a}_+^\dagger|n\rangle\right)^\dagger\cdot(\hat{a}_-|n\rangle) = \qquad\qquad &= \langle n|\cdot(n|n\rangle) \\ (\hat{a}_-|n\rangle)^\dagger\cdot(\hat{a}_-|n\rangle) = \qquad\qquad &= n\langle n\,|\,n\rangle \\ (g_n|n-1\rangle)^\dagger\cdot(g_n|n-1\rangle) = \qquad\qquad &= n\cdot 1 \\ (g_n^*\langle n-1|)\cdot(g_n|n-1\rangle) = \qquad\qquad &= n, \\ g_n^* g_n\langle n-1\,|\,n-1\rangle = \qquad\qquad & \\ |g_n|^2\cdot 1 = \qquad\qquad & \\ |g_n|^2 = \qquad\qquad & \end{aligned}$$

which means

$$f_n = \sqrt{n+1} \quad \text{and} \quad g_n = \sqrt{n}.$$

As a result, the promotion and demotion operators satisfy

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle.$$

Solve this first equation for the $(n+1)$th eigenstate

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}_+|n\rangle$$

and evaluate it for several values of n to find a pattern.

$$\begin{aligned}
|1\rangle &= \frac{1}{\sqrt{1}}\hat{a}_+|0\rangle \\
|2\rangle &= \frac{1}{\sqrt{2}}\hat{a}_+|1\rangle = \frac{1}{\sqrt{2}}\hat{a}_+\left(\frac{1}{\sqrt{1}}\hat{a}_+|0\rangle\right) = \frac{1}{\sqrt{2\cdot 1}}(\hat{a}_+)^2|0\rangle \\
|3\rangle &= \frac{1}{\sqrt{3}}\hat{a}_+|2\rangle = \frac{1}{\sqrt{3}}\hat{a}_+\left[\frac{1}{\sqrt{2\cdot 1}}(\hat{a}_+)^2|0\rangle\right] = \frac{1}{\sqrt{3\cdot 2\cdot 1}}(\hat{a}_+)^3|0\rangle \\
|4\rangle &= \frac{1}{\sqrt{4}}\hat{a}_+|3\rangle = \frac{1}{\sqrt{4}}\hat{a}_+\left[\frac{1}{\sqrt{3\cdot 2\cdot 1}}(\hat{a}_+)^3|0\rangle\right] = \frac{1}{\sqrt{4\cdot 3\cdot 2\cdot 1}}(\hat{a}_+)^4|0\rangle \\
&\vdots \\
|n\rangle &= \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|0\rangle
\end{aligned}$$

Therefore, the normalization constant is

$$A_n = \frac{1}{\sqrt{n!}},$$

and the formula for c_n becomes

$$\begin{aligned}
c_n &= A_n^*\alpha^n c_0 \\
&= \frac{\alpha^n}{\sqrt{n!}}c_0.
\end{aligned}$$

Part (d)

In order to determine c_0, require that the probabilities of measuring E_0, E_1, and so on add to 1.

$$1 = \sum_{n=0}^{\infty}|c_n|^2 = \sum_{n=0}^{\infty}\left|\frac{\alpha^n}{\sqrt{n!}}c_0\right|^2 = \sum_{n=0}^{\infty}\frac{|\alpha^n|^2}{n!}|c_0|^2 = |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{n(2)}}{n!} = |c_0|^2\sum_{n=0}^{\infty}\frac{\left(|\alpha|^2\right)^n}{n!} = |c_0|^2e^{|\alpha|^2}$$

Note that $|\alpha|^2 = \alpha\alpha^*$ is a real number, so $|c_0|^2 = c_0^2$. Therefore,

$$c_0 = e^{-|\alpha|^2/2}.$$

Part (e)

A coherent state $|\alpha\rangle$ is one that satisfies

$$\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle. \tag{1}$$

In parts (c) and (d) it was expanded in terms of the energy eigenstates.

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{2}$$

Apply the demotion operator to both sides and then use equations (1) and (2) on the left.

$$\begin{aligned}\hat{a}_-|\alpha\rangle &= \hat{a}_-\left(\sum_{n=0}^{\infty} c_n|n\rangle\right)\\ \alpha|\alpha\rangle &= \sum_{n=0}^{\infty} c_n(\hat{a}_-|n\rangle)\\ \alpha\sum_{n=0}^{\infty} c_n|n\rangle &= \\ \sum_{n=0}^{\infty} c_n(\alpha|n\rangle) &= \end{aligned}$$

Consequently,

$$\hat{a}_-|n\rangle = \alpha|n\rangle. \tag{3}$$

Now include the wiggle factor in the coherent state to find how it evolves in time (according to the Schrödinger equation).

$$|\alpha(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar}|n\rangle$$

Apply the demotion operator to both sides, noting that because it's in terms of position, the wiggle factor passes right through it.

$$\begin{aligned}\hat{a}_-|\alpha(t)\rangle &= \hat{a}_-\left(\sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar}|n\rangle\right)\\ &= \sum_{n=0}^{\infty} c_n\hat{a}_-\left(e^{-iE_nt/\hbar}|n\rangle\right)\\ &= \sum_{n=0}^{\infty} c_n\left[\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{\partial}{\partial x}+m\omega x\right)\right]\left(e^{-iE_nt/\hbar}|n\rangle\right)\\ &= \sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar}\left[\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{\partial}{\partial x}+m\omega x\right)\right]|n\rangle\end{aligned}$$

Use equation (3) and try to write the right side similar to $|\alpha\rangle$.

$$\begin{aligned}
\hat{a}_-|\alpha(t)\rangle &= \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}\left(\hat{a}_-|n\rangle\right) \\
&= \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}\left(\alpha|n\rangle\right) \\
&= \alpha \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle = \alpha|\alpha(t)\rangle \\
&= \alpha \sum_{n=0}^{\infty} \left(\frac{\alpha^n}{\sqrt{n!}}c_0\right) e^{-i\left[\hbar\omega\left(n+\frac{1}{2}\right)\right]t/\hbar}|n\rangle \\
&= \alpha \sum_{n=0}^{\infty} \left(\frac{\alpha^n}{\sqrt{n!}}c_0\right) e^{-i\omega n t}e^{-i\omega t/2}|n\rangle \\
&= \alpha e^{-i\omega t} \sum_{n=0}^{\infty} \left[\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}c_0\right] e^{i\omega t/2}|n\rangle \\
&= \alpha(t)\left[\sum_{n=0}^{\infty} c_n(t)|n\rangle\right] e^{i\omega t/2}
\end{aligned}$$

Notice the similarity with equation (1). Here, however, the eigenvalue evolves in time, $\alpha(t) = \alpha e^{-i\omega t}$, and

$$c_n(t) = \frac{[\alpha(t)]^n}{\sqrt{n!}}c_0 = \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}e^{-|\alpha|^2/2},$$

and there's a phase factor $e^{i\omega t/2}$.

Part (f)

Calculate the expectation value of x in the state $|\alpha(t)\rangle$.

$$
\begin{aligned}
\langle x\rangle &= \langle \alpha(t)\,|\,\hat{x}\,|\,\alpha(t)\rangle \\
&= \langle \alpha(t)|\cdot[\hat{x}|\alpha(t)\rangle] \\
&= \langle \alpha(t)|\cdot\left[\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)|\alpha(t)\rangle\right] \\
&= \langle \alpha(t)|\cdot\sqrt{\frac{\hbar}{2m\omega}}\left[\hat{a}_+|\alpha(t)\rangle + \hat{a}_-|\alpha(t)\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle \alpha(t)\,|\,\hat{a}_+\,|\,\alpha(t)\rangle + \langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle \alpha(t)\,|\,\hat{a}_+^\dagger\,|\,\alpha(t)\rangle^* + \langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle^* + \langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[2\,\mathrm{Re}\,\langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle\right] \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\left(\sum_{q=0}^{\infty} c_q^* e^{iE_q t/\hbar}\langle q|\right)\hat{a}_-\left(\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle\right) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\left(\sum_{q=0}^{\infty} c_q^* e^{iE_q t/\hbar}\langle q|\right)\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}\,(\hat{a}_-|n\rangle) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\left(\sum_{q=0}^{\infty} c_q^* e^{iE_q t/\hbar}\langle q|\right)\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}\,(\alpha|n\rangle) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\left[\alpha\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q\,|\,n\rangle\right] \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\left[\alpha\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\delta_{qn}\right] \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\left(\alpha\sum_{n=0}^{\infty} c_n^* c_n e^0\right)
\end{aligned}
$$

Therefore, at time t,

$$\begin{aligned}
\langle x \rangle &= \sqrt{\frac{2\hbar}{m\omega}}\,\text{Re}\left(\alpha\sum_{n=0}^{\infty}|c_n|^2\right) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\text{Re}\ \alpha \\
&= \sqrt{\frac{2\hbar}{m\omega}}\,\text{Re}\left(C\sqrt{\frac{m\omega}{2\hbar}}e^{i\phi}\right) \\
&= \sqrt{\frac{2\hbar}{m\omega}}\left(C\sqrt{\frac{m\omega}{2\hbar}}\cos\phi\right) \\
&= C\cos\phi.
\end{aligned}$$

Calculate the expectation value of x^2 in the state $|\alpha(t)\rangle$.

$$\begin{aligned}
\langle x^2 \rangle &= \langle \alpha(t)\,|\,\hat{x}^2\,|\,\alpha(t)\rangle \\
&= \langle \alpha(t)| \cdot \left[\hat{x}^2|\alpha(t)\rangle\right] \\
&= \langle \alpha(t)| \cdot \left[\frac{\hbar}{2m\omega}(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ + \hat{a}_-)|\alpha(t)\rangle\right] \\
&= \langle \alpha(t)| \cdot \left[\frac{\hbar}{2m\omega}(\hat{a}_+\hat{a}_+ + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-)|\alpha(t)\rangle\right] \\
&= \langle \alpha(t)| \cdot \left[\frac{\hbar}{2m\omega}(\hat{a}_+\hat{a}_+ + 2\hat{a}_+\hat{a}_- + 1 + \hat{a}_-\hat{a}_-)|\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle \alpha(t)\,|\,\hat{a}_+\hat{a}_+\,|\,\alpha(t)\rangle + 2\langle \alpha(t)\,|\,\hat{a}_+\hat{a}_-\,|\,\alpha(t)\rangle + \langle \alpha(t)\,|\,\alpha(t)\rangle + \langle \alpha(t)\,|\,\hat{a}_-\hat{a}_-\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle \alpha(t)\,|\,\hat{a}_+\hat{a}_+\,|\,\alpha(t)\rangle + 2\langle \alpha(t)\,|\,\hat{a}_+\alpha\,|\,\alpha(t)\rangle + \langle \alpha(t)\,|\,\alpha(t)\rangle + \langle \alpha(t)\,|\,\alpha^2\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle \alpha(t)\,|\,\hat{a}_+\hat{a}_+\,|\,\alpha(t)\rangle + 2\alpha\langle \alpha(t)\,|\,\hat{a}_+\,|\,\alpha(t)\rangle + \langle \alpha(t)\,|\,\alpha(t)\rangle + \alpha^2\langle \alpha(t)\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle \alpha(t)\,|\,\hat{a}_+^\dagger\hat{a}_+^\dagger\,|\,\alpha(t)\rangle^* + 2\alpha\langle \alpha(t)\,|\,\hat{a}_+^\dagger\,|\,\alpha(t)\rangle^* + (1+\alpha^2)\langle \alpha(t)\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle \alpha(t)\,|\,\hat{a}_-\hat{a}_-\,|\,\alpha(t)\rangle^* + 2\alpha\langle \alpha(t)\,|\,\hat{a}_-\,|\,\alpha(t)\rangle^* + (1+\alpha^2)\langle \alpha(t)\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\langle \alpha(t)\,|\,\alpha^2\,|\,\alpha(t)\rangle^* + 2\alpha\langle \alpha(t)\,|\,\alpha\,|\,\alpha(t)\rangle^* + (1+\alpha^2)\langle \alpha(t)\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[(\alpha^*)^2\langle \alpha(t)\,|\,\alpha(t)\rangle^* + 2\alpha\alpha^*\langle \alpha(t)\,|\,\alpha(t)\rangle^* + (1+\alpha^2)\langle \alpha(t)\,|\,\alpha(t)\rangle\right]
\end{aligned}$$

Therefore, at time t,

$$
\begin{aligned}
\langle x^2\rangle &= \frac{\hbar}{2m\omega}\left[(\alpha^*)^2\langle\alpha(t)\,|\,\alpha(t)\rangle + 2\alpha\alpha^*\langle\alpha(t)\,|\,\alpha(t)\rangle + (1+\alpha^2)\langle\alpha(t)\,|\,\alpha(t)\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[(\alpha^*)^2 + 2\alpha\alpha^* + (1+\alpha^2)\right]\langle\alpha(t)\,|\,\alpha(t)\rangle \\
&= \frac{\hbar}{2m\omega}[(\alpha+\alpha^*)^2+1]\left(\sum_{q=0}^{\infty} c_q^* e^{iE_qt/\hbar}\langle q|\right)\left(\sum_{n=0}^{\infty} c_n e^{-iE_nt/\hbar}|n\rangle\right) \\
&= \frac{\hbar}{2m\omega}[(2\,\mathrm{Re}\ \alpha)^2+1]\left[\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q\,|\,n\rangle\right] \\
&= \frac{\hbar}{2m\omega}[4(\mathrm{Re}\ \alpha)^2+1]\left[\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\delta_{qn}\right] \\
&= \frac{\hbar}{2m\omega}[4(\mathrm{Re}\ \alpha)^2+1]\left(\sum_{n=0}^{\infty} c_n^* c_n e^0\right) \\
&= \frac{\hbar}{2m\omega}[4(\mathrm{Re}\ \alpha)^2+1]\left(\sum_{n=0}^{\infty} |c_n|^2\right) \\
&= \frac{\hbar}{2m\omega}[4(\mathrm{Re}\ \alpha)^2+1] \\
&= \frac{\hbar}{2m\omega}\left[4\left(\mathrm{Re}\ C\sqrt{\frac{m\omega}{2\hbar}}e^{i\phi}\right)^2+1\right] \\
&= \frac{\hbar}{2m\omega}\left[4\left(C\sqrt{\frac{m\omega}{2\hbar}}\cos\phi\right)^2+1\right] \\
&= C^2\cos^2\phi + \frac{\hbar}{2m\omega}.
\end{aligned}
$$

Calculate the expectation value of p in the state $|\alpha(t)\rangle$.

$$
\begin{aligned}
\langle p\rangle &= \langle\alpha(t)\,|\,\hat{p}\,|\,\alpha(t)\rangle \\
&= \langle\alpha(t)|\cdot[\hat{p}|\alpha(t)\rangle] \\
&= \langle\alpha(t)|\cdot\left[i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)|\alpha(t)\rangle\right] \\
&= \langle\alpha(t)|\cdot i\sqrt{\frac{\hbar m\omega}{2}}\left[\hat{a}_+|\alpha(t)\rangle - \hat{a}_-|\alpha(t)\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle\alpha(t)|\hat{a}_+|\alpha(t)\rangle - \langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle\right]
\end{aligned}
$$

Therefore, at time t,

$$\begin{aligned}
\langle p\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle\alpha(t)|\hat{a}_+^\dagger|\alpha(t)\rangle^* - \langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle^* - \langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle\right] \\
&= \sqrt{2\hbar m\omega}\left[\frac{\langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle - \langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle^*}{2i}\right] \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left(\sum_{q=0}^{\infty} c_q^* e^{iE_q t/\hbar}\langle q|\right)\hat{a}_-\left(\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle\right) \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left(\sum_{q=0}^{\infty} c_q^* e^{iE_q t/\hbar}\langle q|\right)\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}\left(\alpha|n\rangle\right) \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left[\alpha\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\langle q\,|\,n\rangle\right] \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left[\alpha\sum_{q=0}^{\infty}\sum_{n=0}^{\infty} c_q^* c_n e^{i(E_q-E_n)t/\hbar}\delta_{qn}\right] \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left(\alpha\sum_{n=0}^{\infty} c_n^* c_n e^0\right) \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left(\alpha\sum_{n=0}^{\infty}|c_n|^2\right) \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\;\alpha \\
&= \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\left(C\sqrt{\frac{m\omega}{2\hbar}}e^{i\phi}\right) \\
&= \sqrt{2\hbar m\omega}\left(C\sqrt{\frac{m\omega}{2\hbar}}\sin\phi\right) \\
&= Cm\omega\sin\phi.
\end{aligned}$$

Calculate the expectation value of p^2 in the state $|\alpha(t)\rangle$.

$$\begin{aligned}
\langle p^2\rangle &= \langle\alpha(t)\,|\,\hat{p}^2\,|\,\alpha(t)\rangle \\
&= \langle\alpha(t)|\cdot\left[\hat{p}^2|\alpha(t)\rangle\right] \\
&= \langle\alpha(t)|\cdot\left[-\frac{\hbar m\omega}{2}(\hat{a}_+ - \hat{a}_-)(\hat{a}_+ - \hat{a}_-)|\alpha(t)\rangle\right]
\end{aligned}$$

Therefore, at time t,

$$\begin{aligned}
\langle p^2\rangle &= \langle\alpha(t)|\cdot\left[-\frac{\hbar m\omega}{2}(\hat{a}_+\hat{a}_+ - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-)|\alpha(t)\rangle\right] \\
&= \langle\alpha(t)|\cdot\left[-\frac{\hbar m\omega}{2}(\hat{a}_+\hat{a}_+ - 2\hat{a}_+\hat{a}_- - 1 + \hat{a}_-\hat{a}_-)|\alpha(t)\rangle\right] \\
&= \langle\alpha(t)|\cdot\frac{-\hbar m\omega}{2}\Big[\hat{a}_+\hat{a}_+|\alpha(t)\rangle - 2\hat{a}_+\hat{a}_-|\alpha(t)\rangle - |\alpha(t)\rangle + \hat{a}_-\hat{a}_-|\alpha(t)\rangle\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[\langle\alpha(t)|\hat{a}_+\hat{a}_+|\alpha(t)\rangle - 2\langle\alpha(t)|\hat{a}_+\hat{a}_-|\alpha(t)\rangle - \langle\alpha(t)\,|\,\alpha(t)\rangle + \langle\alpha(t)|\hat{a}_-\hat{a}_-|\alpha(t)\rangle\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[\langle\alpha(t)|\hat{a}_+^\dagger\hat{a}_+^\dagger|\alpha(t)\rangle^* - 2\langle\alpha(t)|\hat{a}_+\alpha|\alpha(t)\rangle - 1 + \langle\alpha(t)|\alpha^2|\alpha(t)\rangle\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[\langle\alpha(t)|\hat{a}_-\hat{a}_-|\alpha(t)\rangle^* - 2\alpha\langle\alpha(t)|\hat{a}_+|\alpha(t)\rangle - 1 + \alpha^2\langle\alpha(t)\,|\,\alpha(t)\rangle\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[\langle\alpha(t)|\alpha^2|\alpha(t)\rangle^* - 2\alpha\langle\alpha(t)|\hat{a}_+^\dagger|\alpha(t)\rangle^* - 1 + \alpha^2\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[(\alpha^*)^2\langle\alpha(t)\,|\,\alpha(t)\rangle^* - 2\alpha\langle\alpha(t)|\hat{a}_-|\alpha(t)\rangle^* - 1 + \alpha^2\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[(\alpha^*)^2\langle\alpha(t)\,|\,\alpha(t)\rangle - 2\alpha\langle\alpha(t)|\alpha|\alpha(t)\rangle^* - 1 + \alpha^2\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[(\alpha^*)^2\langle\alpha(t)\,|\,\alpha(t)\rangle - 2\alpha\alpha^*\langle\alpha(t)\,|\,\alpha(t)\rangle^* - 1 + \alpha^2\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[(\alpha^*)^2 - 2\alpha\alpha^*\langle\alpha(t)\,|\,\alpha(t)\rangle - 1 + \alpha^2\Big] \\
&= -\frac{\hbar m\omega}{2}\Big[(\alpha^*)^2 - 2\alpha\alpha^* - 1 + \alpha^2\Big] \\
&= -\frac{\hbar m\omega}{2}[(\alpha - \alpha^*)^2 - 1] \\
&= -\frac{\hbar m\omega}{2}\left[-4\left(\frac{\alpha - \alpha^*}{2i}\right)^2 - 1\right] \\
&= \frac{\hbar m\omega}{2}\left[4(\mathrm{Im}\ \alpha)^2 + 1\right] \\
&= \frac{\hbar m\omega}{2}\left[4\left(\mathrm{Im}\ C\sqrt{\frac{m\omega}{2\hbar}}e^{i\phi}\right)^2 + 1\right] \\
&= \frac{\hbar m\omega}{2}\left[4\left(C\sqrt{\frac{m\omega}{2\hbar}}\sin\phi\right)^2 + 1\right] = C^2m^2\omega^2\sin^2\phi + \frac{\hbar m\omega}{2}.
\end{aligned}$$

Now that all the expectation values are known, the standard deviations in position and momentum can be evaluated at time t.

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left(C^2\cos^2\phi + \frac{\hbar}{2m\omega}\right) - (C\cos\phi)^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\left(C^2m^2\omega^2\sin^2\phi + \frac{\hbar m\omega}{2}\right) - (Cm\omega\sin\phi)^2} = \sqrt{\frac{\hbar m\omega}{2}}$$

The uncertainty product for the state $|\alpha(t)\rangle$ is

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}.$$

Therefore, a coherent state stays coherent and continues to minimize the uncertainty product.

Part (g)

The ground state of the harmonic oscillator is a coherent state with eigenvalue 0 because

$$\begin{aligned}
\hat{a}_-|0\rangle &= \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}}\left[\hbar\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)\right] \\
&= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}}\left[\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)\frac{d}{dx}\left(-\frac{m\omega}{2\hbar}x^2\right) + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right] \\
&= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}}\left[\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2}\right)\left(-\frac{m\omega}{\hbar}x\right) + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right] \\
&= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}}\left(-m\omega x e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right) \\
&= \frac{1}{\sqrt[4]{4\pi\hbar^3 m\omega}}(0) \\
&= 0 \\
&= 0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \\
&= 0|0\rangle.
\end{aligned}$$